RESTRICTED LEFT PRINCIPAL IDEAL RINGS*

BY

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ABSTRACT

A ring is an LD-ring if R is left bounded, if R/J is a left Artinian left principal ideal ring for every proper ideal J in R, and if R has finite left Goldie dimension. If R is non-Artinian then R is an order in a simple Artinian ring S. The ideal theory of LD-rings is investigated, and we discuss some conditions under which an LD-ring is an hereditary ring, and some under which an LD-ring is a Noetherian, bounded, maximal Asano order. A central localization of an LD-ring is an LD-ring, and the center of some LD-rings is a Krull-domain.

Recently, Robson [25] studied a class of non-commutative rings that he termed (left, right) Dedekind rings. Bounded Dedekind rings were studied by Michler [19]. Different possible approaches to generalize commutative Dedekind domains are motivated by the various properties of the commutative Dedekind domains that one expects to be inherited by the noncommutative rings. Among the most important properties of commutative Dedekind domains one finds that the set of fractionary ideals forms an abelian group.

The purpose of this paper is to study a special class of rings that we call LD-rings. The hypotheses on a ring to be an LD-ring are very much influenced from the properties of non-commutative Dedekind rings.

The main idea is to verify to what extent are the LD-rings close to Dedekind rings. It turns out that this class is of interest for its own sake. We therefore omit the discussion of the relation between our results and those of Robson, Eisenbud and Robson, and Michler. One of the differences lies in the fact that we impose only one-sided restrictions on the ring, thus we lack symmetry. It seems that if we symmetrize the assumptions we'll end up with bounded Dedekind rings. (See the

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remark to Theorem 3.14, and Theorem 4.4.) We do not specify our results to the commutative case, as those are well-known (e.g. [4], [9], [17], and [26]).

Except for the study of the ideal structure of the LD-rings, we shall also show that in some cases their center is a Krull-domain and under some finite generation restriction we end up with bounded Noetherian, maximal Osano orders over Dedekind domains.

The results we obtain seem to be valuable for the study of rings all proper residue rings of which are quasi-Frobenius.

In Section 1 we discuss the ideal theory, in special rings, in which we investigate a uniqueness property and related consequences.

In Section 2 we list some properties of ideals in an LD-ring, we prove that every ideal can be generated by at most two elements, and that tertiary ideals are primary.

In Section 3 we prove that in an LD-ring R there are no proper idempotent ideals and we discuss some restrictions under which R is a left hereditary ring. This is the case if R is quasi-local ring, or if the fractionary ideals form a group.

In Section 4 we investigate the LD-Ore domain. These rings are always left hereditary. Under a further restriction, namely that R/M is a division ring for every maximal ideal M in R, one obtains a class of rings that are almost commutative rings as one can easily convince himself by looking at their properties. Among other properties, one can localize at maximal ideals, the localization being a local, left Noetherian, left hereditary ring. We also discuss some conditions for these rings to be bounded, Noetherian, maximal Asano orders.

In Section 5 we prove that the center of an almost commutative ring is a Krull-domain, and that being an almost commutative ring is a property inherited by left localizations. We also prove that the center of an LD-ring is a Krull-domain, under an additional restriction, and that being an LD-ring is a property inherited by central localizations.

We close with some related problems.

0. Definitions

All rings are assumed to have an identity, all ideals are two-sided ideals, and all modules are unitary left modules unless otherwise specified. All unspecified one-sided properties of a ring are presumed to hold on both sides, for example, an Artinian (Noetherian) ring is both a left and a right Artinian (Noetherian) ring.

A ring R is a left order in a ring S, if R is a subring of S and every element s in S can be expressed as $s = u^{-1}v$ for a suitable pair of elements u and v from R. We also refer to S as a left quotient ring of R. A ring R is a left Ore domain, if R is a left order in a division ring.

An ideal J in R is a prime ideal if $AB \subset J$, where A and B are ideals, then either $A \subset J$ or else $B \subset J$.

The ring R is a prime ring, if 0 in it is a prime ideal.

A ring R is a left Goldie prime ring if R is a left order in a simple Artinian ring.

The left Goldie dimension of a ring R is bigger than or equal to the integer n, if R contains a direct sum of n ideals. Therefore, the left Goldie dimension of the ring R is finite if the number of terms in a direct sum of ideals of R is bounded.

The left ideal I of R is termed a left essential ideal, if for every left ideal J in R, if $I \cap J = 0$ then $J = 0$.

Note that if R is a prime ring then non-zero ideals are essential, since $IJ \subset I \cap J$ whenever I is a two-sided ideal in R.

An element r in R is regular if r is not a right nor a left zero divisor.

A left fractionary ideal of R in the left quotient ring S of R, is a left R-submodule A of S such that A contains a regular element, and for some regu ar element s in S, $As \subset R$.

An ideal I of R is invertible in the left quotient ring S of R, if there exists in S a left and right fractionary ideal A such that $AI = IA = R$.

Our main interest is to investigate rings that have the following properties:

1) R is left bounded, i.e., every essential left ideal contains a non-zero ideal.

2) R/J is a left Artinian, left principal ideal ring for every non-zero ideal J in R.

3) R is of finite left Goldie dimension.

One may observe that condition (3) is superfluous in case R is a domain.

A ring R satisfying conditions (1), (2) and (3) will be called an LD-ring, and except when otherwise stated, we always assume that R is *not an Artinian* ring.

Let R be a left order in S. Let M be a subset of the set of regular elements of R. Let T be the subset of S consisting of all the elements of form $m^{-1}r$ for some m in M and r in R (in particular this implies that the elements of M have inverses in S). If T is a ring, then T is called a *left localization* of R. If M consists of elements from the center of R we call T a *central localization* of R. Of course central localizations exist at every subset M all of whose elements have inverses in S, whenever M is closed under multiplication.

For the definitions and properties of various types of Asano orders we refer to [1], [12], [19] and [25]. For orders in simple Artinian rings we refer to [11] and [13]. For properties of (Artinian) principal ideal rings we refer to [13].

Let I be a left ideal. Set $J = \{a \mid a \in R$ and $aR \subset I\}$, $K = \{\cap P \mid P$ is a prime ideal in R and $P \supset J\}$, $A = \{a \mid a \in R$ and $aL \subset I$ for some left ideal L that properly contains $I\}$, and $B = \{a \mid a \in R$ such that for every left ideal L that properly contains I there exists a left ideal K between L and I such that $aK \subset I\}$.

The ideal I is a *primary* ideal if $K \supset A$.

The ideal I is a *tertiary* ideal if $B \supset A$.

For properties of primary and tertiary modules (radicals), the relations among them, and for properties of rings for which tertiary modules are primary modules we refer to [24].

R is a local ring if it has a unique maximal two-sided ideal M, and if R/M is a simple Artinian ring.

R is a quasi-local ring if R has only finitely many maximal ideal $M_1, \cdots, M_t$ and if R/M_i is a simple Artinian ring for $i = 1, \cdots, t$.

A ring R is a left generalized uniserial ring if every left component of R has a unique composition series.

A ring R is a left uniserial ring if R is a left generalized uniserial ring, and if R is a direct sum of primary rings.

All the left-sided properties mentioned above can easily be translated to the right-sided properties in the right-left symmetrization. This remark also goes to the rest of this paper, where the right-left symmetric definitions and hypothesis will lead to analogous results.

1. Ideals in special rings

Unless otherwise specified R will denote a prime ring, all of whose non-zero prime ideals are maximal such that:

(i) $\bigcap_{k=1}^{\infty} M^k = 0$ for every maximal ideal M,

(ii) the product of maximal ideals is commutative, and

(iii) every ideal is a finite product of maximal ideals. Observe that all these hypotheses are left-right symmetric.

LEMMA 1.1. *If I, J are ideals in an arbitrary ring R, such that $IJ = JI$ and $I + J = R$, then $IJ = I \cap J$.*

PROOF. Let $i+j=1$, $i\in I$ and $j\in J$. Then for every u in $I\cap J$ we have $u=ui+uj\in JI+IJ$. Therefore $I\cap J\subset IJ$. Obviously $IJ\subset I\cap J$, whence equality holds.

The crucial step towards the unique representation theorem lies upon:

LEMMA 1.2. *$M_1^{k_1}\cap\cdots\cap M_t^{k_t}=M_1^{k_1}\cdots M_t^{k_t}$ whenever M_i are maximal ideals in R such that $M_i=M_j$ iff $i=j$.*

PROOF. Assume $t>1$. Since $I=M_2^{k_2}\cdots M_t^{k_t}\not\subset M_1$ we have $M_1+I=R$. Also obviously $M_1I=IM_1$. Therefore $M_1^2+M_1I=M_1$ which implies the equalities:

$$M_1^2+I=M_1^2+IM_1+I=M_1+I=R, \text{ and of course } M_1^2I=IM_1^2.$$

Proceeding by induction it follows that $M_1^k+I=R$ and $M_1^kI=IM_1^k$. By Lemma 1.1 $M_1^{k_1}I=M_1^{k_1}\cap I$. The proof follows by repeating the procedure on $I=M_2^{k_2}\cdots M_t^{k_t}$, i.e., by induction on t.

As a consequence, we now derive:

COROLLARY 1.3. *Let A be an ideal in R, and M a maximal ideal of R that does not contain A. Then $M^k\cap A=M^kA$.*

PROOF. Since $M\not\supset A$, all one has to observe before applying Lemma 1.2, is that $A=M_1^{m_1}\cdots M_t^{m_t}$ where m_i are positive integers and $M\neq M_i$ for $i=1,\cdots,t$.

COROLLARY 1.4. *Let A be an ideal in R, and M a maximal ideal that does not contain A. If $M^kA=M^{k+1}A$ for any integer k, then $A=0$.*

PROOF. Since $M^kA=M^{k+1}A$, then $\cap_{j=1}^{\infty}M^{k+j}A=M^kA$. On the other hand $\cap_{j=1}^{\infty}M^{k+j}A=\cap_{j=1}^{\infty}(M^{k+j}\cap A)=(\cap_{j=1}^{\infty}M^{k+j})\cap A=0$. Since R is a prime ring it follows that $A=0$.

We therefore obtain the unique representation theorem:

THEOREM 1.5. *Let $M_1^{m_1}\cdots M_t^{m_t}=N_1^{n_1}\cdots N_s^{n_s}$ where $M_i(N_i)$, are maximal ideals, $m_i(n_i)$ are positive integers, and $M_i\neq M_j$ $(N_i\neq N_j)$ whenever $i\neq j$. Then $s=t$, and there exists a permutation π of $(1,\cdots,t)$ so that $M_i=N_{\pi(i)}$ and $m_i=n_{\pi(i)}$.*

As a consequence we obtain the cancellation law, and the divisibility property.

PROPOSITION 1.6. *If $AB=AC$ for A, B, C two-sided ideals and $A\neq 0$, then $B=C$.*

PROPOSITION 1.7. *If $A\subset B$ are ideals, then there exists an ideal C, so that $A=BC$.*

As far as addition is concerned we have

PROPOSITION 1.8.

$$M_1^{m_1} \cdots M_t^{m_t} + M_t^{n_1} \cdots M_1^{n_t} = M_1^{\min(m_1,n_1)} \cdots M_t^{\min(m_t,n_t)},$$

where M_i are maximal ideals and $M_i \neq M_j$ whenever $i \neq j$. We set $A^\circ = R$ for every ideal $A \neq 0$ in R.

PROOF. The proof follows from the equality $M^kA + M^{k+1}A = M^kA$ for every ideal A that is not contained in the maximal ideal M, and from the possibility of inserting additional summands to the sum under consideration: if $m_1 < n_1$ we insert $M_1^{m_1+1}M_2^{m_2}\cdots M_t^{m_t} + \cdots + M^{n_1-1}M_2^{m_2}\cdots M_t^{m_t}$, and in a similar way for $M_2,\cdots,M_t$ (if necessary).

The following will be a useful observation:

LEMMA 1.9. *The condition* (i) *on R may be replaced by the equivalent condition* (i′): *R has no proper idempotent ideals.*

PROOF. (i)$\Rightarrow$ (i′) obviously. (i′)$\Rightarrow$(i): Set $A = \cap_{k=1}^{\infty} M^k$, then $A = M_1^{m_1}\cdots M_t^{m_t}$ and we may assume $M_1 = M$, as $M \supset A$. If $t > 1$, then $M_1 + M_2^{m_2}\cdots M_t^{m_t} = R$. Therefore $M_1^{m_1+1} + A = M_1^{m_1}$, but $M_1^{m_1+1} \supset A$ implies $M_1^{m+1} = M_1^{m_1}$ and this leads to $(M_1^{m_1})^2 = M_1^{m_1}$, therefore $t = 1$. Again $A = M_1^{m_1}$ implies $M_1^{m_1} = (M_1^{m_1})^2$ whence $A = 0$.

Furthermore, if R/M is an Artinian ring for every maximal ideal M then:

PROPOSITION 1.10. *For every non-zero ideal J in R, R/J is a direct sum of matrix algebras S_i over rings T_i that have a unique chain of ideals. $S_i(T_i)$ are semi-primary rings.*

PROOF. Let $J = M_1^{m_1}\cdots M_k^{m_k} = M_1^{m_1} \cap \cdots \cap M_k^{m_k}$ where $M_i \neq M_j$ whenever $i \neq j$ and M_i are maximal ideals of R. We claim that R/J is isomorphic to $R/M_1^{m_1} \oplus \cdots \oplus R/M_k^{m_k}$. Since $M_i^{m_i} + \prod_{j\neq i} M_j^{m_j} = R$ for every i, the natural map f of R into $R/M_1^{m_1} \oplus \cdots \oplus R/M_k^{m_k}$ is an epimorphism, while the kernel of f is precisely J. The ring $S_i = R/M_i^{m_i}$ is a semi-primary ring with nilpotent radical $N_i = M_i/M_i^{m_i}$ and residue ring S_i/N_i which is simple Artinian being isomorphic to R/M_i. Therefore, S_i is a matrix algebra over a semi-primary ring T_i with radical K_i such that T_i/K_i is a division ring. There is a natural one-to-one correspondence between ideals of S_i and T_i. From Theorem 1.5 it follows that there are no ideals between M_i^k and M_i^{k+1} for any k, whence $R/M_i^{m_i} \supset M_i/M_i^{m_i} \supset M_i^2/M_i^{m_i}\cdots$ is the unique sequence of ideals in S_i, and therefore also in T_i we have a unique chain of ideals.

LEMMA 1.11. *Let M be an ideal in an order R in S such that $M^*M = MM^* = R$ for some fractionary ideal M^*. Then M is both a right and a left finitely generated projective module.*

PROOF. As the conditions are left-right symmetric, so are the conclusions, thus it suffices to prove that M is a right finitely generated projective module. To this extent it suffices to show that regarding M as a right module and $\mathrm{Hom}_R(M,R)$ as a left module, then the map $\mu: M \otimes_R \mathrm{Hom}_R(M,R) \to \mathrm{Hom}_R(M.M)$ is an epimorphism, where $\mu(m \otimes f)\,(m') = m \cdot f(m')$ for every pair of elements m, m' in M and every homomorphism f in $\mathrm{Hom}_R(M,R)$, (e.g. [3]). Since Im μ is an ideal in $\mathrm{Hom}_R(M,M)$ it suffices to prove that the identity homomorphism on M is in the image of μ. Since $MM^* = R$ there exists elements $m_1^*, \cdots, m_1^*$ in M^* and $m_1 \cdots m_n$ in M so that $m_1m_1^* + \cdots + m_nm_n^* = 1$. As $M^*M = R$, every element m^* in M^* induces a homomorphism in $\mathrm{Hom}_R(M,R)$ by left multiplication. Identifying $m_1^*, \cdots, m_n^*$ with the homomorphisms they induce we get:

$$\mu(m_1 \otimes m_1^* + \cdots + m_n \otimes m_n^*)(m) = m_1 \cdot m_1^*(m) + \cdots + m_n \cdot m_n^*(n)$$
$$= (m_1m_1^* + \cdots + m_nm_n^*)\ m = m,$$

hence the image of $m_1 \otimes m_1^* + \cdots + m_n \otimes m_n^*$ is the identity and this completes the proof.

2. Ideals in LD-rings

All rings in this section are assumed to be LD-rings, unless otherwise specified. After listing some properties of ideals of arithmetical nature we prove that every left ideal in R can be generated by at most two elements, the first of which is almost arbitrary, and that essential left ideals are tertiary iff they are primary.

From the structure of left Artinian left principal ideal rings (see [13, pp. 75–77]) one derives easily the following properties for a ring R all of whose proper residue rings are left Artinian left principal ideal ring.

1) Every non-zero prime ideal is a maximal ideal.

2) Multiplication of prime ideals is commutative.

3) $M_1^{m_1} \cap \cdots \cap M_k^{m_k} = M_1^{m_1} \cdots M_k^{m_k}$ for every set of distinct prime ideals $(M_1, \cdots, M_t)$ and every set of positive integers $(m_1, \cdots, m_t)$.

4) Every non-zero ideal is the product of maximal ideals, $A = M_1^{m_1} \cdots M_k^{m_k}$, and R/A is isomorphic (as a ring) to the direct sum of $R/M_i^{m_i}$, $i = 1 \cdots, k$.

5) If R is not an Artinian ring, then R is a prime ring. If in addition the proper residue rings are left principal ideal rings then:

6) R/A is a left uniserial ring for every non-zero ideal A.

We start with a direct observation that R is a left Noetherian ring.

LEMMA 2.1. *R is a left Noetherian ring.*

PROOF. Let $I \neq 0$ be an essential left ideal, and let $J \neq 0$ be a two sided ideal that is contained in I. As R/J is a left Artinian ring, it readily follows that every ascending chain of left ideals that contains I as one of its members must become stationary.

Let $I_1 \subset I_2 \subset \cdots$ be an increasing sequence of ideals. Since R is finite dimensional if no I_j is essential, then $\bigcup_{i=1}^{\infty} I_i$ is not essential. Let K be an ideal so that $\bigcup_{i=1}^{\infty} I_i \oplus K$ is an essential ideal. Then $I_1 \oplus K \subset \cdots$ is an increasing sequence of ideals, and for some n $I_n \oplus K$ is an essential ideal. By the argument that started the proof, the last sequence becomes stationary, whence so does the original sequence.

Consequently, R is an order in a simple Artinian ring whenever R is a prime ring. One easily verifies that this is the case under each of the following conditions:

1) R has infinitely many prime ideals,

2) R is not an Artinian ring,

3) the Jacobson radical of R is not nilpotent.

Observe that similar arguments yield:

LEMMA 2.2. *Let R be any ring of finite left Goldie dimension, such that R/I is a left Artinian module for every non-zero essential ideal; then R is a left Noetherian ring.*

That R is a left Noetherian ring is of course also a consequence of:

PROPOSITION 2.2. *Every left ideal is generated by at most two elements.*

PROOF. It suffices to prove the Lemma for essential left ideals. So let I be an essential left ideal. We prove even more then claimed, namely, if r is any regular element in I, then there exists an element $s \in I$ so that $I = Rr + Rs$. To see this, let J be a non-zero ideal in Rr. Such an ideal exists as Rr is an essential left ideal since r is a regular element in R. Therefore, R/J is a left principal ideal ring. In particular I/J is a principal R/J ideal, therefore there exists an element s in R so that $I = Rs + J$. Since $I \supset Rr \supset J$ the result follows.

As for the tertiary left ideals, we have:

PROPOSITION 2.3. *Tertiary essential proper left ideals are primary ideals with a non-zero prime. The non-essential ideals are all primary with zero as their prime.*

PROOF. Let I be an essential tertiary proper left ideal. Since R/I is an Artinian left module, there necessarily exists a maximal ideal M so that $M^m \subset I$ for some integer m. To see this we observe that if $I \supset M^m A$ for some proper ideal A and $M \not\supset A$ and neither $M^m \subset I$ nor $A \subset I$, then as $M^m A = AM^m \subset I$, and as $M^m + A = R$, the assumption that I is a tertiary ideal yields $I = R$: Let $a \in A$ and $a \notin M$. Then $RaR + M = R$, and a consequence of induction is $RaR + M^k = R$ for every integer k, since $MaR + M^{k+1} = M$ implies

$$R = RaR + M = RaR + MaR + M^{k+1} = RaR + M^{k+1}.$$

If for some x in R, $x \notin I$ we have $aRx \subset I$, since $R = RaR + M^m$ then $Rx = RaRx + M^m x \subset RaRx + M^m \subset I$ and this is a contradicton. Consequently I is a primary left ideal with M as its associated prime.

Finally, if I is not an essential ideal, I does not contain an essential ideal, in particular it does not contain a non-zero two sided ideal. Since R is a prime ring, it follows that I is a primary ideal and has 0 as its prime radical.

3. Idempotent ideals and hereditary rings

The main purpose of this section is to prove that LD-rings have no proper idempotent ideals. Essentially, we prove that no proper idempotent ideals exist in a wider class of rings. We also prove that R is a left hereditary ring under each of the following additional hypothesis: (1) R is a quasi-local ring, (2) maximal ideals are left projective, and (3) there are no right fractionary proper idempotent ideals.

We let R be a left order in a simple Artinian ring, such that:

1) Multiplication of maximal ideals is commutative,
2) R is left bounded,
3) The proper residue rings of R are left Artinian, and
4) R is not an Artinian ring.

Let A be a left module. The set of elements in A with an essential left annihilator form a submodule, $T(A)$, the torsion submodule of A. If $A = T(A)$ we say that A

is a torsion module, and if $T(A) = 0$ we say that A is a torsion free module. As R is a left order in a simple Artinian ring we have:

1) For every module A, the module $A/T(A)$ is a torsion free module.

2) Given an exact sequence of left modules $0 \to A' \to A \to A'' \to 0$, then A is a torsion module iff both A' and A'' are torsion modules.

LEMMA 3.1. *Every simple left module is a torsion module.*

PROOF. Let S be a simple left module. If $S \neq T(S)$ then necessarily $T(S) = 0$. Let $s \in S$ be any non-zero element, then its annihilator L is a maximal left ideal, and L is not essential. If J is a left ideal such that $L \cap J = 0$ then necessarily $L \oplus J = R$ and J is a minimal ideal. As R is a prime ring $JL \neq 0$ and as J is a minimal ideal $Jx \neq 0$ implies Jx is a minimal ideal for every x in R. As R is of finite left Goldie dimension this implies that R has an essential left socle K consisting of a finite number of copies of J. That R/K is an Artinian left module follows from R being left bounded, and its proper residue rings being left Artinian. Therefore, R is a left Artinian prime ring, whence a simple ring contradicting the hypothesis on R. Consequently S is a torsion module.

Our next step is to prove that the injective envelop of a simple left module is a torsion module.

LEMMA 3.2. *The injective envelop of a simple left module, is a torsion module.*

PROOF. Let S be a simple left module, and let E be its injective envelop. Let q be any non-zero element in E, then $S \subset Rq$. Let I be the left annihilator of q. If I is not essential, let J be a non-zero left ideal such that $I \cap J = 0$. Then J is isomorphic to $Jq \subset Rq$. As Jq is a non-zero submodule of the injective envelop of S, Jq necessarily contains S. The isomorphism $J \simeq Jq$ yields a minimal ideal in R and a contradiction results as in Lemma 3.1. This completes the proof.

From the commutativity of the multiplication of ideals we obtain:

LEMMA 3.3. *For every pair of distinct maximal ideals M, N and for every pair of finitely generated left R-module K, L such that $M^m K = 0$ and $N^n L = 0$ for some integers m and n, $Ext_R^1(K, L) = 0$.*

PROOF. The proof is by induction on m. If $m = 1$, K is an R/M-module, therefore we may assume it is a simple R-module. Consider any R-module X for which the sequence (*) $0 \to L \to X \to K \to 0$ is exact. Then $N^n MX = 0$ and

$N^nX \neq 0$ since $MX \subset L$ and $M + N^n = R$. Since $N^nX \neq 0$ and $N^nX \not\subset L$, and since $MN^nX = 0$ and K is a simple module this implies that the epimorphism $X \to K$ has a right inverse, whence the exact sequence (*) splits, and this proves that $\mathrm{Ext}^1_R(K, L) = 0$. We proceed by induction on m, assuming the result holds whenever $M^kK = 0$ for $k < m$. Thus let $m > 1$, $M^mK = 0$, and $M^{m-1}K \neq 0$. Consider the commutative diagram with exact rows and columns, and were the maps are the natural homomorphisms:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & M^{m-1}K & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \to & L & \to & Y & \to & K & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \to & L' & \to & Y & \to & K/M^{m-1}K & \to & 0 \\
 & & \downarrow & & & & & & \\
 & & L'' & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

then $L'' \cong M^{m-1}K$, and the induction applies to the exact sequence $0 \to L \to L' \to L'' \to 0$, consequently $L' \simeq L \oplus M^{m-1}K$. Let L_0'' denote the image of L'' in L' under the splitting, $L_0'' \cong M^{m-1}K$. Then we obtain the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & L_0'' & \to & L_0'' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & L' & \to & Y & \to & K/M^{m-1}K & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & L & \to & Y/L_0'' & \to & K^1 & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

therefore $K^1 \approx K/M^{m-1}K$. By induction the exact sequence

$$0 \to L \to Y/L''_0 \to K^1 \to 0 \text{ splits.}$$

Combining both diagrams and the splitting of the last sequence the splitting of the exact sequence $0 \to L \to Y \to K \to 0$ follows from the homomorphisms $L \to L' \to Y \to Y/L''_0 \to L$.

Remark that all that we needed for this lemma is that R/M and R/N are simple Artinian rings and that $MN = NM$.

Let Q be the simple ring in which R is a left order. Then Q is a left injective module and we have an exact sequence:

$$0 \to R \to Q \to Q/R \to 0.$$

In particular $\mathrm{Ext}^1_R(R/M^m, R) \simeq \mathrm{Hom}_R(R/M^m, Q/R)$ for every maximal ideal M and every integer m. We claim that $\mathrm{Ext}^1_R(R/M^m, R) \neq 0$. For let x be any regular element in M^m, then $M^m \supset Rx \supset B$ where B is a non-zero ideal, hence $B = M^m A$ for some ideal A. Therefore $M^m(Ax^{-1}) \subset R$ and if B is a maximal ideal that is contained in Rx then $Ax^{-1} \not\subset R$. Consequently the image of x^{-1} in Q/R generates an R-module U that contains a non-zero submodule V that is annihilated by M^m. The claim $\mathrm{Ext}^1_R(R/M^m, R) \neq 0$ is a consequence of R/M^m being a left Artinian ring with a unique (up to isomorphism) simple module. We thus have:

COROLLARY 3.4. *For every maximal ideal M* $\mathrm{Hom}_R(R/M, Q/R) \neq 0$.

We turn now to the case of idempotent ideals. The infective envelop of R/M as a left module is a torsion module for every maximal ideal M, but we have even more:

LEMMA 3.5. *Let M be a maximal ideal such that $M^m = M^{m+1}$. Then M^m annihilate the injective envelop of R/M.*

PROOF. Let S be a simple module such that $MS = 0$. Let E denote its injective envelop. As E is a torsion module, every finitely generated submodule E' has some power of M as its annihilator. Because, if $NE' = 0$ and $N = M^m B$, where $B \not\subset M$, then unless $M^m E' = 0$ it follows that $BS = 0$ which is impossible as $M + B = R$. In particular, if $E' \neq 0$ then necessarily $E' \neq ME'$. Since $M^m = M^{m+1}$, it is therefore necessary that $M^m E' = 0$ for every finitely generated submodule of E, whence $M^m E = 0$.

Observe that if $M = M^2$ then necessarily R/M is a left R-injective module.

LEMMA 3.6. *Q/R is a direct sum of submodules E_M such that every element of E_M generates a submodule that is annihilated by some power of M. Furthermore, if $M^m = M^{m+1}$, then $M^m E_M = 0$.*

PROOF. Let $E_M = \{q \mid q \in Q/R,\ M^m q = 0 \text{ for some } m\}$. Obviously the subsets E_M form a left (and right) submodule for every M. and the sum ΣE_M is a direct sum. To see that this sum exhausts all of Q/R, let $0 \neq q$ be any element in Q/R and let A be its annihilator. As A is essential let B be a maximal two sided ideal contained in A, then Rq is an R/B-module whence Rq is a direct sum of submodules, each having some power of some maximal ideal as its annihilator. This proves our assertion. That $M^m E_M = 0$ whenever $M^m = M^{m+1}$ is a consequence of Lemma 3.5.

PROPOSITION 3.7. *For every maximal ideal M, $M^m \neq M^{m+1}$ for every integer m.*

PROOF. We derive a contradiction from the assumption $M^m = M^{m+1}$ for some integer m. Since M^m is an essential ideal, it contains a regular element, whence $M^m Q = Q$, therefore $M^m(Q/R) = Q/R$. By Lemma 3.6 $M^m E_M = 0$, however $ME = E_N$ whenever $M \neq N$ because $M + N^n = R$ for every integer n thus if q is in E_N and $N^n q = 0$ and $m + n' = 1$, $n' \in N^k$, then $q = mq$. As $E_M \subset M^m(Q/R)$ the direct sum decomposition of Q/R provided by Lemma 3.6 implies $E_M \subset M^m(E_M) = 0$ and this is in contradiction to Corollary 3.4.

We are now ready to state and prove the main theorem:

THEOREM 3.8. *R contains no proper idempotent ideals.*

PROOF. Let A be a proper idempotent ideal, then $A = M_1^{m_1} \cdots M_n^{m_n}$ where $M_1 \cdots M_n$ are distinct maximal ideals and $m_1, \cdots, m_n$ are positive integers. Also $M_1^{m_1} \cdots M_n^{m_n} = M_1^{2m_1} \cdots M_n^{2m_n}$. As R/A is isomorphic to $R/M_1^{m_1} \oplus \cdots \oplus R/M_n^{m_n}$, and R/A^2 is isomorphic to $R/M_1^{2m_1} \oplus \cdots \oplus R/M_n^{2m_2}$, the assumption $A = A^2$ implies the existence of an idempotent maximal ideal. The uniqueness of the decomposition of R/A (R/A^2) implies an isomorphism between the left Artinian rings $R/M_1^{m_1}$ and $R/M_1^{2m_1}$ which is impossible unless $M_1^{m_1} = M_1^{2m_1}$, but this is impossible by Proposition 3.7.

Some immediate consequences are:

COROLLARY 3.9. *For every proper ideal A in R, $\bigcap_{n=1}^{\infty} A^n = 0$.*

PROOF. Set $B = \bigcap_{n=1}^{\infty} A^n$. If $B \neq 0$ then R/B is a left Artinian ring, whence the sequence $A/B \supset A^2/B \supset \cdots$ becomes stationary. This implies $A^n = A^{n+1} \cdots = A^{2n}$ which is impossible by Theorem 3.8.

Corollary 3.10. *For every left ideal I, $\bigcap_{n=1}^{\infty} I^n = 0$ or else $IR = R$.*

PROOF. Because if $IR \neq R$ then IR is a proper ideal and $I^n \subset I^nR = (IR)^n$. Therefore, $\cap_{n=1}^{\infty} I^n \subset \cap_{n=1}^{\infty}(IR)^n = 0$.

The extent to which the representation of an ideal as a product of maximal ideals in unique may easily be obtained from the result of Section 1. Note that if R is an Artinian local ring this is obvious, while in any other event, no maximal ideal of R is idempotent and the results of Section 1 are applicable:

If $M_1^{m_1} \cdots M_t^{m_t} = N_1^{n_1} \cdots N_s^{n_s}$ where $M_i(N_i)$ are maximal ideals so that $M_i \neq M_j$ $(N_i \neq N_j)$ whenever $i \neq j$, then (i) $s = t$, (ii) one can rearrange $N_1, \cdots, N_t$ say $N_{p(1)} \cdots N_{p(t)}$ so that $M_i = N_{p(i)}$ for $i = 1, \cdots t$, and (iii) $m_i = n_{p(i)}$.

In order to discuss some cases in which R is a left hereditary ring, some remarks about fractionary ideals will be useful.

LEMMA 3.11. *For every maximal ideal M, there exists a fractionary left ideal M^* such that M^* properly contains R, and $MM^* \subset R$.*

PROOF. Take x to be a regular element in M, then there exists an ideal A so that $Rx \subset MA$, and we may assume that A is a maximal ideal with that property. As $Ax^{-1} \not\subset R$ and $M(Ax^{-1}) \subset R$ the result follows.

This also proves

COROLLARY 3.12. *For every maximal ideal M there exists a fractionary right ideal M_* that is a left R-module and that properly contains R, so that $MM_* \subset R$.*

Since $MM_* \supset M$ we either have $MM_* = R$, or else $MM_* = M$, the latter holding only if $M_*^2 = M_*$, if M_* is chosen to be maximal with respect to the property $MM_* \subset R$. Also M_*M is a right fractionary ideal, and if proper right fractionary ideals that are left R-modules are not idempotent then $MM_* = R$ and $(M_*M)^2 = M_*M$ whence $M_*M = MM_* = R$, and M is invertible. Since every ideal is a product of maximal ideal, every ideal is invertible. Consequently,

PROPOSITION 3.13. *If no proper right fractionary ideals that is a left module is an idempotent ideal, then the right fractionary ideals that are left modules form an abelian group that is the free abelian group generated by the maximal ideals. This is the case if every ideal contains an invertible ideal.*

PROOF. Let G denote the free abelian group generated by the maximal ideal, and F any right fractionary ideal that is a left R-module. There exists two sided ideals A and B so that $AF = B$, whence $F = A^{-1}B$ and the result is now immediate. That this is the case if every ideal contains an invertible ideal is a direct consequence of Proposition 1.7 and of the commutativity of the multiplication of ideals in R.

Remark that in this case a left and right R-submodule of Q is a right fractionary ideal iff it is a left fractionary ideal.

That such a ring is necessarily a left hereditary ring is a consequence of the more general theorem:

THEOREM 3.14. *If every maximal ideal is a left projective module, then R is an hereditary ring.*

PROOF. We prove first that all simple modules have projective dimension one, then we use the fact that R is a left Noetherian ring together with the fact that the factor modules R/I are Artinian whenever I is an essential left ideal to conclude that essential left ideals are left projective, and this suffices because every left ideal, if not essential, is a direct summand of an essential ideal.

Let A be a simple R-module. Let $0 \neq a \in A$, then $I = \{x \mid x \in R, xa = 0\}$ is a maximal ideal. If I is not an essential ideal, then there exists an ideal $J \neq 0$ such that $I \cap J = 0$ and as I is a maximal ideal $I \oplus J = R$ and J is a minimal ideal. Since R is a prime ring $JI \neq 0$, but Ji for every element i in I is isomorphic to J, or else $Ji = 0$. Since Goldie-dim $R < \infty$, it follows that R has an essential socle, and being bounded it follows that R is a simple ring, and the result obviously holds.

If R is not a simple ring, then I is essential. Say $I \supset P \neq 0$ for some ideal P in R. Then A is a simple module over the Artinian ring R/P, and as such there exists a maximal ideal M that contains P and is contained in I. Therefore, A is an R/M-module, and for some n, $\sum_{i=1}^{n} A$ is isomorphic as a left R-module to R/M. Since M is a projective ideal, the exact sequence

$$0 \to M \to R \to R/M \to 0$$

implies that l.p. dim $R/M \leqq 1$, whence l.p. dim $A \leqq 1$ and as R is not semi-simple inequality holds, and l.p. dim $I = 0$.

Let K be an essential left ideal, maximal with respect to the property that K is not projective. Such an ideal exists if R is not hereditary. Since R/K is Artinian, there exists a left ideal L, $L \supset K$, such that L/K is a simple left module. Since $L \supset K$, L is an essential left ideal. By the maximality of K, the left ideal L is projective. From the first part of the proof, l.p. dim $L/K = 1$. Therefore, the exact sequence $0 \to K \to L \to L/K \to 0$ implies that K is projective. This contradiction completes the proof.

REMARK. Under the hypothesis of Proposition 3.13 it follows that R is a left hereditary ring from Lemma 1.11 and Theorem 3.14.

We don't know whether in general an LD-ring is left hereditary; however in the quasi-local case we prove it is a left principal ideal ring, whence left hereditary. Also in the quasi-local case, if the fractionary ideals form an abelian group then the proper residue rings are quasi-Frobenius rings.

PROPOSITION 3.15. *If R is a quasi-local ring, and R is not an Artinian ring, then R is a left principal ideal ring.*

PROOF. Let $M_1, \cdots, M_n$ be the set of maximal ideals, and let $J = \bigcap_{i=1}^{n} M_i$. As R is a prime ring J is a non-zero ideal. As $\bigcap_{k=1}^{\infty} J^k = 0$, if y is any element in J, and if $1 - y$ is not a regular element, then $(1 - y)z = 0$ (or $z(1 - y) = 0$) for some non-zero z. As $1 - y$ is invertible in R/J^n for all integers n, it follows that $z \in J^n$, whence $z \in \bigcap_{k=1}^{\infty} J^k = 0$. In particular the Lemma of Nakayama implies $A = Ry$, whenever A is a left ideal and $A = Ry + JA$.

To complete the proof we notice that it suffices to prove that essential ideals are principal, and if A is an essential ideal then $A \supset J^b$ for some b. Therefore A/J^{b+1} being a principal ideal in R/J^{b+1} implies the existence of an element y in R such that $A = Ry + J^{b+1} \subset Ry + JA \subset A$. Since every ideal is principal, and two-sided ideals are essential, a two-sided ideal is generated by a regular element, whence no proper left fractionary ideal that is a right module is an idempotent ideal.

There naturally arises the problem as to whether every ideal is invertible in the quasi-local case, since being principal each ideal has a right inverse that is a right fractionary ideal. However, if every ideal has an inverse then the set of fractionary ideals form an abelian group, generated by the ideals.

PROPOSITION 3.16. *If R is a quasi-local ring, if R is not an Artinian ring, and if every ideal has an inverse than all the proper residue rings of R are QF-rings.*

PROOF. By Proposition 3.15 R is a principal ideal ring. If M is any non-zero ideal, say $M = Rm$, then $M^* = m^{-1}R$. If $M^*M = R$ then $m^{-1}Rm \subset R$ or $Rm \subset mR$, whence $Rm = mR$. In particular, every ideal is also a principal right deal. To study the proper residue rings of R it is necessary and sufficient to study the residue rings R/M^m for some integer m, and maximal ideal M, but this ring is a left and right uniserial ring, whence a quasi-Frobenius ring.

4. Ore domains

The main result concerning LD-Ore domains is that they are left hereditary. Under the additional condition on R/M being a division ring for every maximal ideal M we discuss localizations, and its relation with being a bounded Dedekind domain.

Let R be an LD-ring that is a left Ore domain. Then every proper left ideal is essential since every element of R is regular. We assume as always that R is not an Artinian ring unless otherwise specified. As a consequence of R being an LD-ring, for every ideal $M \neq 0$ in R there exists an element x in R so that $M = Rx + M^2$. We start with a remark about the ideal Rx.

LEMMA 4.1. *Let $M = Rx + M^2$, then $Rx \supset MA$ and $M \not\supset A$.*

PROOF. Let B be the maximal ideal contained in Rx. As $M \supset Rx$, we have $B = MA$. If $M \supset A$, then $B = M^2C$. The equation $M = Rx + M^2$ yields $MC \subset Rx + M^2C \subset Rx$, and as MC strictly contains B, this contradicts the maximality assumption on B.

A crucial consequence is:

COROLLARY 4.2. *Let $M = Rx + M^2$, then R/Rx contains a simple R/M-module as a direct summand.*

PROOF. By Lemma 4.1 we have $Rx \supset MA$ and $M \not\supset A$. If $A = R$, then $M = Rx$ and we are done. If $A \neq R$, then R/Rx is an $(R/MA =)$ $R/M \oplus R/A$-module. Since $M \not\supset A$ we must have $A \not\supset Rx$, therefore $(A+Rx)/Rx$ is a non-zero submodule and $M(A+Rx)/Rx = 0$, whence $(A+Rx)/Rx$ is a non-zero R/M-module. Since $AM = A \cap M$, then $R/Rx = (A+Rx)/Rx \oplus (M+Rx)/Rx$ and the Corollary follows.

In particular, it follows now that every maximal ideal is a left projective module, or since R is not Artinian and is left bounded, we have equivalently:

PROPOSITION 4.3. *Every simple left module has projective dimension one.*

PROOF. Consider the exact sequence, for any non-zero element x in R:

$$0 \to Rx \to R \to R/Rx \to 0.$$

Since R is an Ore domain Rx is a projective left module being isomorphic to R.

Therefore, l.p. dim $R/Rx \leqq 1$. That inequality holds is a consequence of R having no proper idempotent elements. In particular if we choose x to be such that $M = Rx + M^2$, it follows from Corollary 4.2 that a simple R/M-module has projective dimension one, whence l.p. dim $R/M = 1$. Therefore l.p. dim $R/M = 1$ for every maximal ideal. Since for every simple left module U there exists a maximal deal M such that $MU = 0$ it follows that U is an R/M-module, therefore l.p. dim $U = 1$.

The main theorem may now easily be derived:

THEOREM 4.4. *R is a left hereditary ring.*

PROOF. From Proposition 4.3 it follows that maximal ideals are left projective modules, and the theorem follows from theorem 3.14.

For the rest, we add the condition: R/M is a division ring for every maximal ideal. It easily follows that every left ideal is then necessarily a right ideal.

Under this additional conclusion, the result that R is a left hereditary ring can be proved in a direct and shorter way, namely: Considering an element x in M^m that is not in M^{m+1}, then $Rx = M^m A$, $M \not\supset A$ and the exact sequence $0 \to Rx \to R \to R/M^m \oplus R/A \to 0$ implies that l.p. dim $R/M^m = 1$ whence M^m is projective for every maximal ideal M and every integer m. Let I be any left ideal, then $I = M_1^{m_1} \cdots M_n^{m_n}$ whence the exact sequence $0 \to I \to R \to R/M_1^{m_1} \oplus \cdots \oplus R/M_n^{m}$ $\to 0$ implies that I is a projective left module since

$$\text{l.p. dim}(R/M_1^{m_1} \oplus \cdots \oplus R/M_n^{m_n}) = 1.$$

The next Lemma gives us a measure to the extent to which R is a commutative ring, and it turns out to be useful in particular in the study of the localizations, and consequently in the study of the center.

LEMMA 4.5. *For every pair of elements a and b in R, and for every integer n there exists invertible elements u_n and v_n in R so that $(ab)^n = u_n(ba)^n = v_n a^n b^n$. (The elements u_n and v_n may depend upon a and b).*

PROOF. By Lemma 4.3 and by Propositions 3.2 and 3.3 it follows that the product of left ideals is commutative, and $Rx \supset xR$ for every $x \in R$. Therefore, for every element y in R, we have $Rxy \subset (Rx)(Ry) = (Ry)(Rx) = R(yR)x \subset Ryx$ and the inverse inclusion is obtained in a similar way, whence $Rxy = Ryx$. An immediate consequence is the existence of an invertible element r in R so that $rxy = yx$. Therefore we obtain by using induction that $R(ab)^n$ equals the product of $(Ra)^n$ and $(Rb)^n$. whence $R\pi$ where π is the product of $2n$ terms, half of them equals a and the other half equals b. The conclusion is now immediate.

Since R/M is a division ring for every maximal ideal M in R, the set $\{x \mid x \in R, \forall c \in R\, xc \in M \Rightarrow c \in M\}$ is identical with the set $R - M$. We next study the possibility of localizing at M, and the properties of the resulting local ring:

THEOREM 4.6. *The set $L(M) = \{a^{-1}b \mid a, b \in R,\ a \notin M\}$ is a ring that is a left Noetherian local, hereditary ring for every maximal ideal M in R. The maximal ideal of $L(M)$ is generated by M as a left ideal in $L(M)$.*

PROOF. Let $a^{-1}b,\ x^{-1}y \in L(M)$, then for some invertible element z in R, $ax = zxa$ and $ax \notin M$. Therefore $a^{-1}b + x^{-1}y = (ax)^{-1}[zxb + ay] \in L(M)$. Furthermore, there exists an invertible element $w \in R$ so that $xb = wbx$, whence $(a^{-1}b)\,(x^{-1}y) = a^{-1}(x^{-1}wb)y = (xa)^{-1}(wby) \in L(M)$. Thus $L(M)$ is a ring. The way $L(M)$ is defined, it is clear that any non-zero left ideal in $L(M)$ intersects R in a non-zero ideal. In particular, if $P \neq 0$ is a prime ideal in $L(M)$, then $P \cap R \neq 0$ is a prime ideal in R. If $P \cap R = N \neq M$ then there exists an element n in N not in M, whence $n \in P$ but in $L(M)$ n is invertible, which is impossible. We claim: $P = L(M)M$. Of course $P \supset L(M)M$. Therefore we will be done if we can prove that $L(M)M$ is a maximal ideal in $L(M)$. First, observe that $L(M)M$ is a two-sided ideal. For let $a^{-1}b \in L(M)$, then $ma^{-1}b \in L(M)M$ for every $m \in M$, because of the existence of invertible elements v_1, v_2 in R so that $ma = v_1am$, $mb = v_2bm$ wherefrom it follows that $ma^{-1}b = (v_1a)^{-1}(v_2b)m \in L(M)M$. Next, let $a^{-1}b \in L(M)$, then for some $v \in R$ we have $av + m = 1$ for an appropriate element m in M. In $L(M)$ we have thus $v + a^{-1}m = a^{-1}$. If $a^{-1}b \notin L(M)M$ then $vb + a^{-1}mb = a^{-1}b$ implies that $vb \notin M$. In particular it results the existence of an element $s \in R$ such that $svb - 1 \in M$, therefore $s(a^{-1}b) - 1 \in L(M)M$. Since every element $a^{-1}b \notin L(M)M$ is invertible modulo $L(M)M$, it follows that $L(M)M$ is a maximal ideal. We claim that $L(M)M$ is a principal left ideal in $L(M)$. For let $M = Rx + Ry$, $Rx = M^mA$, and $Ry = M^nB$ with $M \not\supset A$, $M \not\supset B$ and $m \geqq n$. Then $Bx = M^mAB$ and $Ay = M^nAB$, whence $Bx \subset Ay$. Let $b^* \in B$, $b^* \notin M$, then b^* is invertible in $L(M)$. The existence of an element a^* in A satisfying $b^*x = a^*y$ in R implies $x = (b^*)^{-1}a^*y$ in $L(M)$, and $L(M)M = L(M)y$. Since $(L(M)M)^n = L(M)y^n$, and $L(M)y^n \cap R = M^n$, it follows that $\bigcap_{n=1}^{\infty}(L(M)M)^n \cap R = \bigcap_{n=1}^{\infty}((L(M)M)^n \cap R)$ $= \bigcap_{n=1}^{\infty}M^n = 0$, whence $\bigcap_{n=1}^{\infty}(L(M)M)^n = 0$. This necessarily implies that $L(M)$ is a left Noetherian principal ideal domain, whence a left hereditary ring.

We end this section pointing out some conditions that force R to be a Noetherian, bounded, maximal Asano order in the division ring D.

Since Rx is a two-sided ideal in R for every element x in R, it follows by Proposition 1.7 that for every ideal A there exists a subset A' of D that is both a left and a right R-module and such that $AA' = R$. In particular this implies that no proper left fractionary ideal F is an idempotent ideal, whenever F is a right R-module. To see this, let A be an ideal such that $FA = B \subset R$, then if $F = F^2$ it follows that $B^2 = (FA)(FA) = F(AB) = F(BA) = F(FA)A = (FA)A = BA$, whence $B = A$. But $FA = A$ implies upon multiplication on the right by A' that $F = R$.

Whether it is possible to have a right fractionary ideal G that is an idempotent ideal, and that is a left R-module we do not know. However, if this is not so Proposition 3.13 assures us that the fractionary ideals of R form an abelian group. Under the further restrictions imposed on the ring in this section it is just natural to obtain a stronger result:

PROPOSITION 4.7. *R is a Noetherian, bounded, maximal Asano order, under each of the following conditions:*

1) *The fractionary ideals form a group.*
2) *R is a right order in D.*
3) *D is a finite dimensional algebra over its center C.*

PROOF. 1) Let m be any non-zero element of R. As Rm is a two-sided ideal $Rm \supset mR$ whence $m^{-1}R \supset Rm^{-1}$, and as (Rm) $(m^{-1}R) = R$ necessarily $(m^{-1}R) = (Rm)^{-1}$, whence $(m^{-1}R)$ $(Rm) = R$ or $Rm = mR$. In particular, every right ideal is a left ideal, and furthermore every set of generators for an ideal as a left ideal may also serve as a generating set for it as a right ideal. Whence R is a Noetherian, bounded, maximal Asano order.

2) Let x, a be a pair of non-zero elements in R, then $x^{-1}ax$ is an element in D. Since R is a right order we must have elements y and b in R so that $x^{-1}ax = yb^{-1}$, consequently $axb = xy$. But $R(axb) = (Ra)(Rx)(Rb) = (Rx)(Ry) = R(xy)$, therefore $(Ra)(Rb) = Ry$ or $y = tab$ for some invertible element t in R. In particular $yb^{-1} = ta$ is an element of R. Hence $x^{-1}ax$ is an element of R for every element a in R. But $x^{-1}Rx \subset R$ implies $Rx \subset xR$. As Rx is a two sided ideal $Rx = xR$, and again R is a Noetherian, bounded domain. That R is a maximal Asano order follows from $Rx = xR$ for every x in R, which in turn implies that every ideal contains an invertible ideal. The result follows from Proposition 3.13.

3) Let $g_v(u) = v^{-1}uv$, and assume $CR = CS_1$, where S_1 denotes the image of R in D under g_v. For every element s in S_1 we have two representations:

$$s = \sum_{i=1}^{n} r_i(x_i^{-1}y_i) \tag{1}$$

for elements x_i, y_i, r_i in R such that $x_i^{-1}y_i$ is an element of C, and

$$s = v^{-1}uv \tag{2}$$

for some element u in R. Since $x_i^{-1}y_i$ is an element of C then $x_iy_i = y_ix_i$ for $i = 1, \cdots, n$. In particular, $s = \sum_{i=1}^{n} r_i(y_ix_i^{-1}) = tc^{-1}$, where $0 \neq c = a_1x_1 = \cdots = a_nx_n$ and $t = \sum_{i=1}^{n} r_ia_iy_i$, thus both t and c are elements of R. Comparing both expressions for s we have $s = v^{-1}uv = tc^{-1}$. This yields $R(uvc) = R(vt)$ and consequently $R(uc) = Rt$. Therefore, $t = buc$ for some (invertible) element b of R, and $s = tc^{-1} = bu$ is an element of R. Hence $CR = CS_1$ implies $R = S_1$.

From Lemma 4.5 it follows that vrv^{-1} is an element of R for every pair of elements v and r of R ($v \neq 0$). Consequently the image of R under g_v contains R. As g_v is an isomorphism on D, then if S_1—the image of R under g_v—properly contains R, then necessarily S_2—the image of S_1 under g_v—properly contains S_1. Inductively, we obtain a strictly increasing sequence of rings S_i where S_{n+1} is the image under g_v of S_n. Correspondingly there results the increasing sequence $CR \subset CS_1 \subset CS_2 \subset \cdots$. But D being a finite dimensional C algebra this last sequence becomes eventually stationary, say $CS_n = CS_{n+1}$. But the argument made above proves that in this case $S_n = S_{n+1}$ which is a contradiction. Consequently $R = S_1 = v^{-1}Rv$, or $Rv = vR$ for every element v in R, and in particular R is a right order in D, whence the result follows from case (2).

Remark that the results of [19] and the first two parts of the last proof establish the equivalence:

PROPOSITION 4.8. *The following are equivalent:*

1) *The fractionary ideals of R form a group.*

2) *R is a right order in D.*

3) *R is a Noetherian, bounded, maximal Asano order.*

Under each of these equivalent conditions every right ideal is a left ideal, the ring $L(M)$ consists precisely of the set of elements $\{ba^{-1} \mid a$ and b are elements of R, $a \notin M\}$, and $R = \cap L(M)$, the intersection taken over all maximal ideals of R, and the proper residue rings of R are quasi-Frobenius rings (e.g. [19]).

If $Z(S)$ denotes the center of the ring S, then since $R = \cap L(M)$ it necessarily follows $Z(R) = \cap Z(L(M))$, consequently $Z(R)$ is a Krull-domain, as $Z(L(M))$ is a discrete valuation ring for every ideal M (Theorem 4.6 and [21]).

Professor P. M. Cohn kindly informed me that for every Krull-domain Z it is possible to construct a ring R so that (1) R is an LD-Ore domain (2) R/M is a division ring for every maximal ideal M in R, and (3) $Z(R) = Z$.

In the next section we discuss the center of R, and we show that $Z(R)$ is always a Krull-domain.

5. Centers and localizations

Let Z be the center of R, let C be the center of S, let D be the field of quotient of Z, let $T = CR$ and let S be a simple Artinian ring. Let R be an order in S, then we have natural embeddings of Z in D, and D in C.

We prove that an LD-ring has as its center a Krull-domain in each of the following cases: (1) $T = S$ and $C = D$, and (2) R is an Ore domain such that R/M is a division ring for every maximal ideal M.

As far as localizations are concerned, we prove that being an LD-ring is a property that is inherited by central localizations, and that being an LD-Ore domain such that R/M is a division ring for every maximal ideal M is a property that is inherited by left localizations.

Observe that if A is a left (right) R ideal, then CA is a left (right) T ideal.

If R is a left (right) order in S then R is a right (left) order in S whenever $T = S$. To see this, remark that if $x^{-1}y$ is an element of C, for x and y in R, then $xy = yx$.

Unless otherwise stated, we assume for the rest of this section that $T = S$ and, $C = D$, and R is an LD-ring, that is not an Artinian ring.

As an essential ideal contains a regular element, and since for every regular element x in R, there exist elements y in R and z in Z such that $x^{-1} = yz^{-1} = z^{-1}y$, it follows that $Rz = zR \subset Rx \cap xR$. In particular, every ideal contains an invertible ideal. Applying Proposition 3.13 we have:

LEMMA 5.1. *The fractionary ideals form a group.*

The same observation also leads to:

LEMMA 5.2. *R is a right bounded ring.*

LEMMA 5.3. *Proper residue rings of R are right Artinian.*

PROOF. From Lemma 5.1, the ideals of R are finitely generated as right R-modules. This implies a finite composition series for R/M^m as a right R-module for every maximal ideal M and for every integer m, consequently R/A is right Artinian for every proper ideal A.

PROPOSITION 5.4. *R is a right Noetherian ring.*

PROOF. As R has a finite right Goldie dimension, as R is right-bounded, and as the proper residue rings of R are right Artinian, the result follows (see Lemmas 2.1, 2.2).

Consequently R is a bounded, Noetherian ring whose fractionary ideals form a group. By Lemma 1.11 and Theorem 3.14 R is hereditary. Whence R is a bounded, Noetherian, maximal Asano order and by [19] there exists local over-rings $K(M)$ of R for each maximal ideal M, such that $K(M)$ is a principal ideal ring, and $R = \bigcap_M K(M)$. Therefore, $Z = \bigcap_M Z(K(M))$.

LEMMA 5.5. *$Z' = Z(K(M))$ is a discrete valuation ring.*

PROOF. Since the ideals of $K(M)$ are all of the form $M^m K(M)$ for some integer m, every non-invertible element u of Z' generates an ideal $K(M)u = M^n K(M)$ for a suitable integer n. The fractionary ideals of $K(M)$ form a group [e.g. [19]), whence if v is any other element of Z' then either $u^{-1}v$ or uv^{-1} lies in Z'. Consequently, Z' is a valuation ring. It is discrete since $K(M)u = M^n K(M)$, and since $K(M)u \cap Z' = Z'u$.

PROPOSITION 5.6. *Z is a Krull-domain, and the family of its essential valuations is induced by the maximal ideals of R.*

Under some finiteness conditions, Z is a Dedekind domain [see 1].

We turn now to the Ore domain case, under the further restriction on R/M to be a division ring. In this case, we do not impose any relations between T and S, nor between C and D. Instead of using $K(M)$ we use $L(M)$ (see Theorem 4.6), but we do not know if $R = \bigcap_M L(M)$. However,

LEMMA 5.7. $Z = \bigcap_M Z(L(M))$.

PROOF. Let c be any element of C such that $c \in Z'$, where we set

$$Z' = \bigcap_M Z(L(M)).$$

Then for every maximal ideal M there exists a pair of elements x and y in R so that x is not in M, $xy = yx$, and $c = x^{-1}y$. Fix M, x and y. Let N be any maximal ideal that contains x. Then $c = a^{-1}b$ where $a, b \in R$, $a \notin N$. Consequently $a^{-1}b = x^{-1}y = yx^{-1}$ or $bx = ay$. Since $Ra \not\subset N$, if $Rx \subset N^n$ then $Ry \subset N^n$. This holding for every maximal ideal N that contains x, it follows that $Rx \supset Ry$ (as both are two-sided). In particular, $y = dx$ for some d in R, and $c = yx^{-1} = d \in R$. Of course this implies that c lies in Z. That $Z \subset Z'$ is obvious, whence $Z = Z'$ as stated.

We therefore obtain

PROPOSITION 5.8. *The center of an LD-Ore domain R, for which R/M is a division ring for every maximal ideal, is a Krull-domain whose family of essential valuations is induced by the maximal ideals of R.*

On the way to study the center of R some localizations turn out to be useful. Those localizations ($L(M)$ and $K(M)$) both inherited the properties of the ring R. It seems natural to investigate the general localization of R in S.

PROPOSITION 5.9. *Let V be a left localization of an LD-Ore domain R, for which R/M is a division ring for every maximal ideal M or R. Then such is V.*

PROOF. Let I be any left ideal of V, then $I \cap R = J$ and $VJ = I$. That V is an Ore-domain is obvious, and that V is left bounded is a consequence of R being such together with the inclusion $Ax \supset xA$ for every ideal A in R, which implies $x^{-1}A \supset Ax^{-1}$. These yield the conclusion that VJ is a two-sided ideal in V whenever J is a two-sided ideal in R. To verify that proper residue rings of V are left Artinian, left principal ideal ring. Let I be a two-sided ideal in V, then J is a two-sided ideal in R. For every left ideal B in V that contains I, let $B' = B \cap R$ then $VB' = B$. However, for some b in B', $B' = Rb + J$ whence $B = Vb + I$, therefore V/I is a left principal ideal ring. Since to a chain of left ideals in V that contains I there corresponds $(1-1)$ a chain of left ideals in R that contains J, it follows that V/I is left Artinian. Finally, let M be a maximal two-sided ideal in V, and set $N = M \cap R$. Were not N a maximal ideal of R, there would have existed elements a and b in R, neither of them in N while $aRb \subset N$. As $ax^{-1} = x^{-1}\alpha a$ for some invertible element α in R, and for element x in R, it follows that $aVb \subset M$ which is a contradiction since M is a prime ideal. Since N is a maximal left ideal in R, M is necessarily a maximal left ideal of V, whence V/M is a division ring for every maximal ideal M of V.

If V is a left localization at the multiplicative set X, $X \subset R$, and if $AX \supset XA$ for every left ideal then VA is a two-sided ideal in V whenever A is a two-sided ideal in R. Following the proof of Proposition 5.9 we have:

PROPOSITION 5.10. *Let V be a left localization of an LD-ring R at a multiplicative set X of R for which $AX \supset XA$ for every essential left ideal, then V is an LD-ring.*

Note that if $X \subset Z$ then the hypothesis holds, whence:

PROPOSITION 5.11. *A central localization of an LD-ring is an LD-ring.*

6. Epilog

We wish to close with some after-thoughts that arise from the results obtained in this paper. The main one is of course:

1) Is a prime LD-ring a left hereditary ring?

Closely related to this question in view of Theorem 4.4 is:

2) Is being an LD-ring a Morita invariant property, more specifically, is a prime LD-ring Morita equivalent to an LD-Ore domain?

Another question that may help in getting an answer to question 1 is:

3) Are the proper residue of an LD-ring quasi-Frobenius rings?

The real origin of this question lies in the fact that this is the case in the examples of LD-rings that we know.

Finally, a question that is merely a matter of curiosity:

4) Let R be an LD-ring that is a left order in D_n—the $n \times n$ matrix ring over the division ring D. Let R/M be $D(M)_n$, where $D(M)$ is a division ring, for every maximal ideal M. Is R Morita equivalent to an LD-Ore domain R' for which R'/M' is a division ring for every maximal ideal M' of R'?

7. Note

Since this paper was written we proved theorem 4.4 for an arbitrary prime LD-ring (Some rings are hereditary rings [28]). I wish to thank Professor C. Robson for pointing out the following remarks:

(1) The results on the center can be derived from results of G. M. Bergman and P. M. Cohn in Center of 2 firs and Hereditary rings (to be published).

(2) In his Ph.D. Dissertation, T. Lenegan obtained more results on localizations in prime LD-rings, and he has a shorter way to prove Theorem 3.8.

(3) Let $K = k(y)$ be a field of rational functions in one variable over a field k. Let x be a variable over K, that commutes with elements of k, and such that $xy = y^2x$. Then $K\{x\}$ is an LD-domain which is not a right Goldie ring.

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